16+ Scholarship — Specimen Paper — Mathematics 2

1. (i) By Pythagoras' Theorem, $DC = \sqrt{20^2 - 4^2} = \sqrt{384} = 19.6$ cm.

   (ii) Since $BC = 4$ cm, the vertical height of M "above" the base AB is 2 cm, so the area of ABM is $\frac{1}{2} \times AB \times 2 = 19.6$ cm^2.

2. (i) $9u = u + 2a$, and so $2a = 8u$, so $a = 4u$.

   (ii) Manipulating,

$$\begin{aligned} y &= \frac{ax+b}{cx+d} \\ (cx+d)y &= ax+b \\ cxy+dy &= ax+b \\ cxy-ax &= b-dy \\ x(cy-a) &= b-dy \\ x &= \frac{b-dy}{cy-a} \end{aligned}$$

3. (i) $\frac{81}{16}$ and $\frac{243}{32}$ (powers of $\frac{3}{2}$).

   (ii) $\frac{9}{8}$ and $\frac{11}{16}$.

   (iii) $\frac{n^2}{n+1}$.

4. (i) Alice shells $\frac{1}{2}$ the peas in an hour. Ben shells $\frac{1}{3}$ of the peas in an hour. Together they shell $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ of the peas in an hour, and so they shell all the peas in $\frac{6}{5}$ hours, or 1 hour and 12 minutes.

   (ii) In that time, Alice shells $\frac{6}{5} \times \frac{1}{2} = \frac{3}{5}$ of the peas. Alternatively, since the times Alice and Ben take to shell the same amount of peas is in the ratio $2 : 3$, the number of peas they shell in a fixed time is in the ratio $3 : 2$, and so Alice will always shell $\frac{3}{5}$ of the peas that the two of them shell in any time period.

5. (i) After 3 years I will have £$400 \times 1.0225 \times 1.0275^2 =$ £431.80 to the nearest penny.

   (ii) After 36 months the friend will have £$400 \times 1.002^{36} =$ £429.83 to the nearest penny.

6. (i) $\frac{4}{9} \times \frac{4}{9} = \frac{16}{81}$.

   (ii) To score 17, I must choose 8 and 9 in some order. The probability of doing this is $\frac{1}{9} \times \frac{1}{9} + \frac{1}{9} \times \frac{1}{9} = \frac{2}{81}$.

   (iii) Since all white balls have odd numbers on them, if both balls chosen are white, then the sum of the numbers of them must be even! Thus we are simply interested in the probability that both balls are white, which is $\frac{5}{9} \times \frac{5}{9} = \frac{25}{81}$.

7. (i) $3 \diamond (-3) = 3^2 + (-3)^2 = 18$.

   (ii) $2 \diamond (3 \diamond 4) = 2 \diamond (3^2 + 4^2) = 2 \diamond 25 = 2^2 + 25^2 = 629$.

(iii)

$$\begin{aligned} x \diamond x &= 3x+9 \\ x^2+x^2 &= 3x+9 \\ 2x^2-3x-9 &= 0 \\ (2x+3)(x-3) &= 0 \end{aligned}$$

and hence $x = -\frac{3}{2}, 3$.

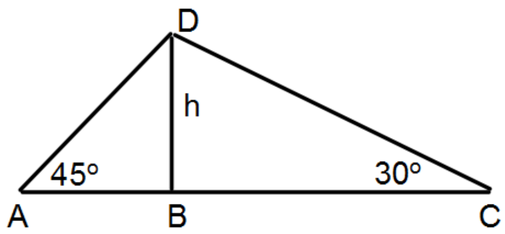
8. We can write

$$x^2+8x-4 = (x+4)^2-16-4 = (x+4)^2-20$$

Since $(x+4)^2$ is a perfect square, $(x+2)^2 \geq 0$ for all real x, and so $x^2+8x-4 \geq -20$. The minimum possible value is -20, and this occurs when $x+4=0$, so when $x=-4$.

9. The right-angled triangle ABD is isosceles, and hence $AB = h$. Considering the right-angled triangle BCD, we deduce that

$$\frac{h}{BC} = \tan 30^\circ$$



Thus

$$\begin{aligned} AB+BC &= 120 \\ h+\frac{h}{\tan 30^\circ} &= 120 \\ h\tan 30^\circ + h &= 120\tan 30^\circ \\ h\left(\tan 30^\circ+1\right) &= 120\tan 30^\circ \\ h &= \frac{120\tan 30^\circ}{\tan 30^\circ+1} \end{aligned}$$

and so $h = 43.9$ m (to 3 SF).

10. (i) We have

$$\frac{x^2-y^2}{(x-y)^2} = \frac{(x-y)(x+y)}{(x-y)^2} = \frac{x+y}{x-y}$$

(ii) Putting everything over the common denominator 15,

$$\frac{2x-4}{15}+\frac{x}{5}-\frac{x-1}{3} = \frac{(2x-4)+3x-5(x-1)}{15} = \frac{1}{15}$$

11. Eliminating x from two of the equations, and then eliminating y from one of these, shows that

$$\begin{array}{r} x+2y-z=6 \\ 2x-y+z=-1 \\ 3x+5y-7z=20 \end{array} \quad \Leftrightarrow \quad \begin{array}{r} x+2y-z=6 \\ -5y+3z=-13 \\ -y-4z=2 \end{array} \quad \Leftrightarrow \quad \begin{array}{r} x+2y-z=6 \\ y+4z=-2 \\ 23z=-23 \end{array}$$

from which we deduce that $z=-1$, $y=2$ and $x=1$. The solution is $(1,2,-1)$.

12. (i) $\frac{4}{5}N$ start out from school, $\frac{4}{5}N-8$ are left after the service station, $\frac{1}{2}\left(\frac{4}{5}N-8\right)=\frac{2}{5}N-4$ do not get lost, and so a total of $\frac{2}{5}N-4-5=\frac{2}{5}N-9$ make it back to school.

(ii) We need to solve the equation

$$\begin{aligned} \tfrac{2}{5}N-9 &= \tfrac{1}{4}N \\ \tfrac{2}{5}N-\tfrac{1}{4}N &= 9 \\ \tfrac{3}{20}N &= 9 \end{aligned}$$

and so $N=60$.

13. (i) We have

$$\begin{aligned} x &= 1+p^{-1} \\ x^2 &= 1+2p^{-1}+p^{-2} \\ x^3 &= 1+3p^{-1}+3p^{-2}+p^{-3} \\ x^4 &= 1+4p^{-1}+6p^{-2}+4p^{-3}+p^{-4} \end{aligned}$$

and hence the equation becomes

$$\begin{aligned} 2x^4+x^3-6x^2+x+2 &= 0 \\ \begin{array}{c} 2+8p^{-1}+12p^{-2}+8p^{-3}+2p^{-4}+1+3p^{-1}+3p^{-2}+p^{-3} \\ -6-12p^{-1}-6p^{-2}+1+p^{-1}+2 \end{array} &= 0 \\ 9p^{-2}+9p^{-3}+2p^{-4} &= 0 \\ 9p^2+9p+2 &= 0 \end{aligned}$$

(ii) We solve

$$\begin{aligned} 9p^2+9p+2 &= 0 \\ (3p+1)(3p+2) &= 0 \end{aligned}$$

and hence $p=-\frac{1}{3},-\frac{2}{3}$. Substituting these values into the formula $x=1+p^{-1}$ gives the solutions $x=-2,-\frac{1}{2}$. Thus $x+2$ and $2x+1$ are factors of the quartic polynomial, and we calculate

$$2x^4+x^3-6x^2+x+2 = (2x+1)(x^3-3x+2) = (2x+1)(x+2)(x^2-2x+1) = (2x+1)(x+2)(x-1)^2$$

Thus the complete solution of $2x^4+x^3-6x^2+x+2=0$ is $x=-\frac{1}{2},-2,-1$ (twice).